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Temperley–Lieb algebras and the long distance properties of statistical mechanical models

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Abstract. We write down the regular representation of the Temperley-Lieb algebra $T_k(q)$ in the basis of reduced words on the k Temperley-Lieb generators. When $k = 2n-1$, representations of these generators may be used to construct the transfer matrices for statistical mechanical models on an n-site wide lattice. We show that the generically irreducible representation of $T_{2n-1}(q)$ responsible for the unique free energy in such models may be restricted to the regular representation of $T_{n-1}(q)$. We give equivalent forms for the regular representation, derived from lattice models, which manifest its indecomposable structure when k and q are such that $T_k(q)$ is semisimple. We hence generalise to obtain the structure of $T_k(q)$ when not semisimple.

A transfer matrix built with generators in the regular representation gives the long distance properties of all possible such models. We show how to find the semisimple quotient algebra which gives these long distance properties when $T_k(q)$ itself is not semisimple. We hence classify the long distance properties of these statistical mechanical models.

1. Introduction

The n-site layer transfer matrices for a wide range of statistical mechanics models including the q-state Potts model, the percolation problem, the ice-type model (Baxter 1982) and the critical Andrews-Baxter-Forrester (ABF) model (1984) may be written in the form

$$T = \left(\prod_{i=1}^{n} (1 + xU_{2i-1})\right)\left(\prod_{i=1}^{n-1} (x + U_{2i})\right). \tag{1}$$

Here the matrices $\{U_i\}$ give a representation of the Temperley-Lieb algebra (Temperley and Lieb 1971) (hereafter referred to as TL) $T_{2n-1}(q)$ defined by the relations

$$\begin{aligned} &U_iU_i = \sqrt{q}\,U_i \\ &U_iU_{i\pm1}U_i = U_i \\ &U_iU_j = U_jU_i \qquad |i-j| > 1 \end{aligned} \tag{2}$$

with q in general an arbitrary scalar parameter labelling the model, and x is a temperature variable. The precise representation involved depends on the model and the boundary conditions. Many of these models are of particular interest because their critical field theory limits exhibit conformal symmetry (Friedan *et al* 1984 (FQS), Kuniba *et al* 1986 (KAW), Pasquier 1987).

The long distance correlation properties of such a model are determined by the eigenvalues of the transfer matrix. To see this consider the correlation function for two separated local measurements

$$\langle \mathcal{O}_0 \mathcal{O}_j \rangle \equiv \langle \alpha | T^m \hat{\mathcal{O}} T^j \hat{\mathcal{O}} T^{M-m-j} | \beta \rangle / \langle \alpha | T^M | \beta \rangle \tag{3}$$

where $\langle \alpha |$ and $\langle \beta | = | \beta \rangle^\dagger$ are vectors corresponding to some boundary conditions, which may be written

$$\langle \alpha | = \sum_i C_{\alpha i} \langle v_i | \qquad \text{(where } \langle v_i | T = \langle v_i | \lambda_i \text{ with } \lambda_i > \lambda_j \Rightarrow j > i)$$

such that $c_{\alpha 0} \neq 0$, and $\hat{\mathcal{O}}$ is an operator of the form

$$\sum_\alpha |\alpha\rangle \sum_\gamma D_\gamma \langle \gamma | = \sum_\alpha |\alpha\rangle \sum_i C_i \langle v_i | \tag{4}$$

where D_γ is the result of a measurement $\mathcal{O}$ on state γ and $C_i = 0$ for $i = 0, \ldots, k_{\mathcal{O}} - 1$. We then have

$$\langle \mathcal{O}_0 \mathcal{O}_j \rangle \xrightarrow[j \to \infty]{} \exp(-j/\xi_{\mathcal{O}}) \tag{5}$$

where $\xi_{\mathcal{O}}^{-1} = \ln(\lambda_0) - \ln(\lambda_{k_{\mathcal{O}}})$.

These eigenvalues in turn are given by the roots of the characteristic polynomial of T, at least for finite n. In general the calculation of eigenvalues is thus greatly facilitated by the factorisation of the characteristic polynomial into its irreducible components (Ahlfors 1979). Each of these gives an analytically distinct part of the spectrum. A big step in this direction is the identification of irreducible representations in the model representation R, say, of the Temperley–Lieb algebra, since this gives a partial block diagonalisation of T. The remaining step is the incorporation of symmetries such as translation invariance (see, for example, Schultz *et al* 1964).

In a recent paper (Martin 1988a) we showed how to identify the irreducible content of the Potts model representation, where attention is restricted to a particular semisimple quotient algebra considered by Jones (1983). In many of the interesting cases, however, the relevant algebras are not semisimple and an arbitrary representation cannot be written as a direct sum of irreducibles. In general this problem may be overcome by quotienting an algebra A by its radical J_A (Anderson and Fuller 1974) which is the maximal nilpotent double-sided ideal subalgebra. Because this radical is nilpotent it contains only non-propagating contributions to the transfer matrix, which do not affect the bulk properties. The quotienting leaves a semisimple 'top' algebra A/J. In the Potts case the algebra is further quotiented to leave an algebra with only unitarisable representations (i.e. representations in which the $\{U_i\}$ are self-adjoint). This combined quotienting is relatively simple (see later), but unfortunately it has the effect of eliminating some propagating contributions, i.e. long distance properties, in more general models. For example, with $q = 1$ it leaves only a one-dimensional representation whereas, as we shall see, there are an infinite number of distinct irreducible representations in the thermodynamic limit. These are physically relevant, for example, in the percolation problem (see Baxter 1982). We will discuss the minimal quotienting procedure which leaves all of them intact.

Now, it is a general theorem that the regular representation contains copies of all the irreducible representations of an algebra (semisimple or otherwise) as simple modules. In fact the regular representation may be written as a direct sum of indecomposable projective left modules (indecomposable representations). There is a bijection between indecomposable projectives P and irreducibles S which sends P to P/JP. Furthermore, the multiplicity of P in the regular representation is the dimension of the associated irreducible. In the present paper we thus start by writing down the regular representation for the Temperley-Lieb algebra. We will give various bases for this representation, which will allow us to discuss the structure of the algebra, and its top, and hence label possible long distance properties of statistical mechanical models.

For various reasons, interest in TL algebras is currently very high. In addition to their relevance for statistical mechanics and conformal field theory they have an intimate connection with knots and braids (see below). These are now the focus of physicists' attention in themselves, following recent remarks of Witten (1988) on their role in string theory. For this reason we will take the oppotunity here to provide the tools for a detailed analysis of any TL algebra. The regular representation has as a basis a spanning set of elements of the algebra. Specifically we may use the set of distinct words in $\{U_i\}$. We show how these operators may themselves be arranged to exhibit the structure of the algebra (i.e. as primitive idempotents and so on). This necessarily involves conveying a fair number of new results. In the interests of brevity we will restrict ourselves to illustrative examples in the text, and leave proofs, where helpful, to the appendix.

2. Preliminary technical remarks

We will begin by making a few observations on $T_k(q)$. The elements

$$t_i^{\pm 1} = 1 - e^{\pm\theta} U_i$$

(where $e^{\theta} + e^{-\theta} = \sqrt{q}$) obey the braid relations (Temperley 1986), plus

$$t_i^2 = (1 - e^{2\theta}) t_i + e^{2\theta}$$

and

$$\{t_i, t_{i+1}\} = t_i + t_{i+1} + t_i t_{i+1} t_i - 1.$$

The algebra has an involution $U_i \to U_{k-i+1}$ (reflection on the lattice) and further automorphisms

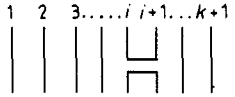
$$U_i \to S U_i S^{-1} = U_{i+1} \tag{6}$$

(duality/translation) where

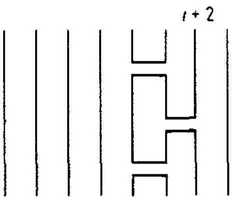
$$S = \prod_{i=1}^{k} t_i$$

so that, defining $U_0 = S U_k S^{-1}$, we have $U_1 = S U_0 S^{-1}$. We will use the notation W^T for the element obtained by writing the generators in element W in reverse order. Note that (2) is invariant under the reversal T.

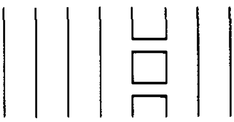
A faithful diagrammatic realisation of words (products of generators U_i) in $T_k(q)$ is obtained as follows. The generator U_i is drawn as



Generators are composed by connecting the bottom of the diagram for the first factor with the top of the diagram for the second, so $U_i U_{i+1} U_i = U_i$ is



(note that we can pull straight the line starting in the $i+2$ position); and $U_i U_i = \sqrt{q}U_i$ is



Provided that we interpret closed loops as contributing a factor $\sqrt{q}$ then these diagrams replace any word with an equivalent word (under the relations (2)) of shortest length in the generators, with an appropriate scalar factor. We will call the set of shortest words the reduced words.

3. The defining representation

The defining representation (TL) has $U_i \in \mathrm{End}(V^{\otimes k+1})$, where the space V is two dimensional. Ordering the product of vector spaces (and labelling them in order) as $V_1 \otimes V_2 \otimes V_3 \otimes \ldots$ we have

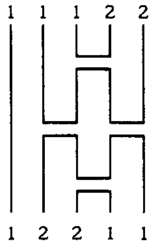
$$U_i = 1 \otimes 1 \otimes 1 \otimes \ldots \otimes \begin{pmatrix} 0 & & & \\ & x & 1 & \\ & 1 & y & \\ & & & 0 \end{pmatrix} \otimes \ldots \otimes 1 \otimes 1$$

where $x = 1/y = \exp(\theta)$ and the 4×4 matrix acts on $V_i \otimes V_{i+1}$. Note that the ordering is arbitrary. This representation breaks up into blocks as follows. Writing $V = (1, 2)$ we see that non-zero off-diagonal entries in U_i just take 12 ($\in V_i \otimes V_{i+1}$) $\leftrightarrow 21$. Writing an arbitrary basis state as $a_1 a_2 \ldots a_{k+1} (a_i \in 1, 2)$ we then note that only states with the same number of 1's are mixed. We will write $\#(1)$ for the number of 1's.

Consider the subspace with $\#(1) = \#(2)$ or $\#(1) = \#(2)+1$. This gives a faithful representation of the algebra. We demonstrate this by showing that each distinct linear combination of reduced words in the algebra is represented by a distinct matrix. To do this we introduce a partial order on the reduced words and show that each word gives a unit contribution to at least one matrix element with the property that only words immediately above it in the partial order also contribute to that matrix element.

The matrix elements are labelled by pairs of basis states each of which are some permutation of (for example, with $k=9$) 1111122222. The partial order is as follows.

Associate with the diagram for word w_1 a basis pair with the property that, on labelling each endpoint by either 1 or 2, the endpoints connected in moving down the diagram are labelled the same, while those connected in the same layer are labelled 12 (reading from left to right) in the top layer and 21 in the bottom layer. The labelling of downward connections here is not, in general, completely determined by $\#(1)$. The freedom arises simply because the representation is not irreducible. It will not affect our arguments. Considering the basis pair so formed, the partial order is the transitive extension of the relation $w_1 < w_2$, where $w_1 < w_2$ if the corresponding basis pair for w_2 is the same up to the interchange of connected pairs of labels within a layer of the w_2 diagram. For example, the diagram corresponding to $U_3U_2U_4U_3$ has basis pair $11122 \to 12211$ in this construction, i.e.



The diagram corresponding to $U_3U_2U_4U_3U_1$ has $11122 \to 21211$, which is the same up to the interchange of the first two numbers in the second state, so $U_3U_2U_4U_3 < U_3U_2U_4U_3U_1$. Note that every sequence $w_1 < w_2 < \ldots < w_p$ ends with a word for which the basis pair supports no equivalent up to interchange. In our example the final word is $U_3U_2U_4U_1U_3U_2$, giving $11122 \to 22111$. This pair can only be connected one way, regardless of the freedom to interchange.

For each matrix element associated with a word as above, i.e. for each such pair of basis states, we can move from the first state to the second by a sequence of moves of the form $a_1a_2a_3a_4\ldots 12\ldots \to a_1a_2a_3a_4\ldots 21\ldots$. For example, $11122 \to 12211$ is achieved by $11122 \to 11212 \to 12112 \to 12121 \to 12211$. From the definition of the representation we note that the matrix element of a corresponding word (where correspondence is defined as replacing each transposition by a U_i in the equivalent position, i.e. $U_3U_2U_4U_3$ in the above example) is 1. If other words have non-vanishing contributions in this element then they are necessarily higher in the partial order, from the definition of the moves and the representation. To see this, note that other possible moves associated with non-vanishing matrix elements are $12 \to 12$ and $21 \to 21$. Since there is a unique final word in any partial order sequence we can read off the contribution of this word directly from the appropriate matrix element, subtract its contribution to other matrix elements accordingly, and hence iteratively read off the contribution of every word in the sequence.

Note that subspaces with $\#(1) = \#(2) + p$ must have at least p lines travelling from the top layer to the bottom. By the same argument they thus give a faithful representation of the algebra quotiented by words whose diagrams have fewer than p such lines.

Given these results, once we have established the structure of the Temperley-Lieb algebra (see below) we could simply read off the structure of tensor product representations of the corresponding quantum groups! We will carry out this procedure explicitly elsewhere.

4. The regular representation

Consider the set of possible 'connectivities' (Blote and Nightingale 1982) of n sites, that is the subset of distinct partitions of n sites numbered $1, \ldots, n$ which satisfy the condition that if sites a and c are connected (in the same partition) and sites b and d are connected and $a<b<c<d$ then all four are connected. Number this set from $1, \ldots, C_n$ and form $2n-1$ C_n-dimensional matrices as follows:

$$
\begin{aligned}
(U_{2i-1})_{jk} &= q^{(\delta_{jk})/2} && \text{if disconnecting the } i\text{th site takes connectivity } j \text{ to } k \\
&= 0 && \text{otherwise} \\
(U_{2i})_{jk} &= q^{(\delta_{jk})/2} && \text{if connecting the } i\text{th and } (i+1)\text{th sites takes } j \text{ to } k \\
&= 0 && \text{otherwise.}
\end{aligned} \tag{7}
$$

These matrices form a representation of the Temperley-Lieb algebra for $2n-1$ operators. Up to a similarity transformation this is the representation we have called the Whitney representation (Martin 1986). The transfer matrix in this case is that of the n-site wide square lattice Whitney polynomial (Baxter 1982). When $q=4$ the Temperley-Lieb algebra is a quotient algebra of the group algebra of the permutation group on $2n$ objects. The above representation is then an irreducible isomorphic to Young's semi-normal representation for tableau of two equal rows. We note by continuity that the representation is thus irreducible at all but a discrete set of q values (see also Hoefsmit 1974).

The basis of C_n possible connectivities of n sites has a one-to-one mapping onto the set of distinct operator products (words) on $n-1$ operators (including the identity). To see this, arrange the n sites into a (vertical) line (so that there are $n-1$ gaps) and label any $n-1$ consecutive sites and gaps from 1 to $n-1$ (for definiteness starting at the topmost site, say). There is then a unique connectivity (partition) P_1 having $[(n+1)/2]$ disconnected components arranged in such a way that imposing a connection across any numbered gap changes the connectivity and disconnecting a numbered site changes the connectivity. In our case this is the partition $\{(1, 2n-1), (3, 2n-3), \ldots\}$. The connectivity P_1 maps onto the identity. The connectivities associated with the other operator products are obtained by reading the operator product from right to left and, in that order, connecting the adjacent sites if the operator number corresponds to a gap, and disconnecting the site if the operator corresponds to a site.

Repeated occurrences of a single operator only change the connectivity once; the effect of the sequence $U_i U_{i\pm 1} U_i$ is equivalent to the effect of U_i; and the order of operation for $U_i U_j$ with $|i-j|>1$ is unimportant (cf the defining relations). Note also that, as required, the number of distinct connectivities C_n (given by

$$C_1 = 1 \text{ and } C_n = (4n-2)C_{n-1}/(n+1)$$

or equivalently by $C_n = (2n)!/(n!(n+1)!) = 1, 2, 5, 14, \ldots$ (Blote and Nightingale 1982)) is equal to the number of distinct words (Jones 1983). Now if we consider the set of matrices defined above, but restrict ourselves to the subset of $n-1$ consecutive operators, we see that the action of each matrix gives precisely the action of the corresponding operator on each element of the set of distinct operator products. This is therefore the regular representation of the $(n-1)$-operator algebra.

Now the matrices defined above give a representation of the full $(2n-1)$-operator algebra corresponding to the linear transformations induced in a certain minimal left ideal LR. The ideal LR is generated by left multiplication of elements of the algebra on R, where

$$R = \prod_{\text{odd } i} U_i. \tag{8}$$

To see this note that the set of connectivities above have a one-to-one mapping onto the set of distinct operator products (words) ending in R for $2n-1$ operators. The argument proceeds as before, except that all sites and gaps are numbered, and the 'seed' partition corresponding to the product R itself is that with no sites connected. It is easy to see that (for q non-zero) the operator $Rq^{-n/2}$ (n is the number of odd operators) is idempotent. This idempotent is also primitive. Thus provided the algebra is semisimple the representation is irreducible (Hamermesh 1962). In any case the restriction of this representation to the subalgebra generated by $n-1$ consecutive generators gives the regular representation of the subalgebra.

Before proceeding to analyse this representation it is useful to define another representation, which is a generalisation of the Andrews-Baxter-Forrester representation discussed in KAW, i.e. that derived from the transfer matrix for the n-site diagonal layer critical $(r-1)$-state Andrews-Baxter-Forrester model. Consider the set of sequences of $2n+1$ numbers $s_1, \ldots, s_{2n+1}$ such that $s_1 = \alpha$, $s_{2n+1} = \alpha + m$ and $|s_i - s_{i+1}| = 1$ (so that m is an even integer). The number of sequences in a set is the binomial coefficient $\binom{2n+1}{n+m/2}$. Numbering the sequences from 1 and introducing the notation s_{ij} for the ith element in the jth sequence, we may write down the following matrices:

$$\begin{aligned} (U_{i-1})_{jk} &= \frac{[\sin(s_{ij}\pi/r)\sin(s_{ik}\pi/r)]^{1/2}}{\sin(s_{i+1j}\pi/r)} \qquad && \text{if } s_{lj} = s_{lk} \text{ for all } l \neq i \text{ and } s_{i+1} = s_{i-1} \\ &= 0 && \text{otherwise} \end{aligned} \tag{9}$$

for $i = 2, \ldots, 2n$. These matrices obey the Temperley-Lieb relations with $q = 4\cos^2(\pi/r)$. It is not necessary to restrict to integer r, and the generalisation to even numbers of generators is straightforward. Generically (r not integer) the transformation $\alpha \to \alpha + \beta$ is just a similarity transformation. These representations are equivalent to the blocks exhibited in the defining representation (above), and also discussed by Martin (1986) where they are termed the initially reduced Temperley-Lieb representations. To be precise, m is the surfeit of up arrows over down arrows in the ice-model lattice layer. We can define

$$\sigma_i^x = 1 \otimes 1 \otimes \ldots \otimes \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \otimes \ldots \otimes 1 \qquad \text{and} \qquad \sigma_i^y = 1 \otimes 1 \otimes \ldots \otimes \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \otimes \ldots \otimes 1$$

where the unit matrices are 2×2, the Pauli matrix appears in the ith position in the product, and the sub-basis states are directions of an arrow on a bond of the 'medial' lattice. The objects

$$U_j = \sigma_j^y \sigma_{j+1}^y (1 - \sigma_j^x \sigma_{j+1}^x) + e^{\theta}(1 + \sigma_j^x)(1 - \sigma_{j+1}^x) + e^{-\theta}(1 - \sigma_j^x)(1 + \sigma_{j+1}^x)$$

may then be simultaneously block diagonalised such that the bases for the blocks have a fixed excess of up arrows over down arrows. These blocks are then isomorphic to the above representations. In the case $q = 4$ they correspond to permutation representations of the permutation group in the canonical basis (Robinson 1961).

Considering the form of U_i in the cases $\alpha = p$ (p integer) we note that here the subset of sequences obeying $s_i > 0$ also provide the basis for a representation. In particular if $p = 1$ we recover the generically irreducible representations discussed, for instance, by Jones (1983). By noting the action of U_i on basis elements we may then associate primitive idempotents in the algebra with the various sequences (see also Temperley 1986, Goodman *et al* 1987). For example, the action of U_i on a sequence is: (i) to multiply by $\sqrt{q}$ if $s_i, s_{i+1}, s_{i+2} = 1, 2, 1$; (ii) to mix with the sequence identical in all but the $(i+1)$th entry for $s_i, s_{i+1}, s_{i+2} = k, k\pm 1, k$ (integer $k>1$); and (iii) to multiply by zero otherwise. Since the sequence 121212 ... 1212345 ... $(m+1)$ is unique in having no subsequences of the form $k, k+1, k$ we start by writing down an idempotent I_m for this.

Define $\text{idem}_c[0] = \text{idem}_c[1] = 1$, and $\text{idem}_c[d+2]$ $(d>-1)$ as the unique idempotent operator which is left and right orthogonal to $U_{1+2c}, \ldots, U_{1+2c+d}$ and contains no other generators. Provided $r'>b$ (where r' is the denominator of r expressed in its lowest form for r rational, and is infinite otherwise) the operator $\text{idem}[b] = \text{idem}_0[b]$ may be constructed recursively as

$$\text{idem}[b] = \text{idem}[b-1](1-\kappa_b U_{b-1})\,\text{idem}[b-1] \tag{10}$$

where $1/\kappa_b = \sqrt{q} - \kappa_{b-1}$ with $\kappa_2 = 1/\sqrt{q}$, i.e.

$$\kappa_b = \frac{\sin((b-1)\pi/r)}{\sin(b\pi/r)} = \frac{\sinh((b-1)\theta)}{\sinh(b\theta)}.$$

The general case may then be obtained by translation. For example

$$\text{idem}_c[2] = 1 - q^{-1/2} U_{1+2c}$$

$$\text{idem}_c[3] = 1 + (U_{1+2c}U_{2+2c} + U_{2+2c}U_{1+2c} - \sqrt{q}(U_{1+2c} + U_{2+2c}))/(q-1)$$

$$\begin{aligned}\text{idem}_0[4] = 1 &- ((q-1)(U_1+U_3) + qU_2)/\sqrt{q}(q-2)\\ &+ (U_1U_2 + U_2U_1 + U_2U_3 + U_3U_2)/(q-2) - (U_3(U_1U_2 + U_2U_1 - \sqrt{q}U_1)\\ &+ (U_1U_2 + U_2U_1 - \sqrt{q}U_1)U_3)/\sqrt{q}(q-2) - (q-1)U_1U_3/q(q-2)\\ &+ \left(\frac{(1-\sqrt{q}U_2)}{\sqrt{q-1}}\,\frac{U_1U_3}{q}\,\frac{(1-\sqrt{q}U_2)}{\sqrt{q-1}}\right)(q-2)^{-1}\end{aligned}$$

and so on (note the symmetry under $U_i \to U_{b-i}$ in $\text{idem}_0[b]$). For $2n-2+a$ operators, where $a=2$ or 1 and $\mu = (m-a+1)/2$, a suitable choice for the initial idempotent is then

$$I_m = \left(\prod_{i=1}^{n-\mu} q^{-1/2} U_{2i-1}\right) \text{idem}_{n-\mu}[2\mu + a - 1]. \tag{11}$$

The idempotent I_m is primitive and orthogonal to I_n $(m \neq n)$. Subsequent primitive idempotents are obtained inductively. That for a sequence with $s_i, s_{i+1}, s_{i+2} = k, k+1, k$ is given by that for a sequence differing only in the $(i+1)$th position left and right multiplied by

$$\sqrt{\kappa_k \kappa_{k+1}}(1 - U_i/\kappa_k). \tag{12}$$

We see that this construction defines a partial order on the basis states labelled by a given m with those of the form 12121234567 (for example) the unique first and 12345678987 (in this case) the unique last.

Provided that none of the κ_i vanish, we thus have the matrix algebras indicated by Jones (1983) (see also Temperley 1986). Off-diagonal elements (called elementary operators, after the corresponding elementary matrices) are obtained by only multiplying on the left (or right) of an idempotent by (12). The proof of the above statements is by a straightforward induction. Some explicit illustrative examples are given later.

Of greater interest, however, are cases where some κ_i do vanish. These occur at the 'Beraha' q values ($r \in \mathbb{Z}$, see Baxter 1982, 1987) which give conformal models at criticality (FQS, Cardy 1986). Here idem$[r-1]$ is well defined and, noting (6), we see in particular that $\{\text{idem}_c[r-1]=0$ for all $c\}$ is the set of quotient relations satisfied by the 'unitarisable' semisimple quotient algebra associated with the Potts model (consider (9) with $r \in \mathbb{Z}$). We will return to these Beraha cases shortly.

The physical significance of the above formalism is that we can construct an operator appropriate for any observable in the sense of (5) from these elementary units. If the algebra representation associated with a model is not irreducible, then the operator $\hat{\mathscr{C}}$ in (3) is not always expressible as an element of an algebra. We note from the above, however, that there is always an operator in the algebra with the same long distance correlation properties (i.e. the same k_r as $\hat{\mathscr{C}}$). In other words, the formalism above allows systematic access to the various analytically disjoint sectors of the spectrum of T. For an example of a physical observable which *is* expressible as an element of the algebra, consider the spin-spin correlation for two sites within a layer for the $q=2$ (Ising) case. Then for the sites at positions p and $p+j$ we have

$$\hat{\mathscr{C}} = \prod_{i=p}^{p+j-1} (\sqrt{2}U_{2i}-1).$$

Our formalism allows us to decompose this operator into elementary components. The expectation value can then be calculated in the (generally much smaller) irreducible subspaces. Examples of the use of this calculational tool are given by Martin (1988b). Of course the Schultz *et al* (1964) solution of the Ising model is another striking example.

Note that with $p=1$, $m=0$ we have a representation of dimension C_n based on the same primitive idempotent as the Whitney representation (compare (8) with (11)). For integer r this representation has some divergent matrix elements if $r<2n+1$, but its trace is finite (and equal to $C_{n-1}\sqrt{q}$—see later) for all r. The representation may be made well defined by regarding r as a variable and making certain similarity transformations before taking r to its integer value. We will give examples shortly.

If r is not rational then the representation is irreducible and thus equivalent to the Whitney representation. The restriction to $n-1$ operators in this case is again, therefore, the regular representation of the subalgebra. In this basis, however, the irreducible content of the regular representation is easy to see as follows. The basis states are associated with distinct possible configurations of ABF variables, s_i, in a lattice layer, subject to the constraint that the boundary variables are set to $s_1=s_{2n+1}=1$. More generally, as we will see, if $s_1=1$ then the possible values of s_{2n+1} (s_{2n-2}) give bases for the various irreducible representations of the $2n-1$ (respectively $2n$) operator algebra. In the ABF model there is the additional constraint that $s_i<r'$. We have discarded this (but see KAW). From the definition of the representation we note that, on restriction to the *first* $n-1$ operators, basis elements corresponding to configurations which differ in any of the last $n+1$ variables are not coupled. Thus the representation breaks up into blocks labelled by the $(n+1)$th and subsequent variables. But it is the $(n+1)$th variable alone which *distinguishes* these representations (see (9)). By reflection symmetry the multiplicity of each irreducible representation is thus equal to its dimension. This describes the regular representation for a semisimple algebra. The Bratelli diagram (Jones 1983) follows immediately, and hence the trace of any generator can be computed. From (1) and Perron's theorem (Bellman 1960) we then identify the $m=0$ representation as the one responsible for the largest eigenvalue of the transfer matrix T (the free energy, see Baxter (1982)) for real temperatures.

In other words we have found a basis for the regular representation in which the irreducible content is manifest. Unfortunately in the non-semisimple cases (r integer) the similarity transformations required to render the matrix elements well defined obscure this picture. We must proceed more carefully. Let us consider an example which will illustrate the point, and also allow us to introduce some useful terminology. Adopting the shorthand $y_\pm = \sin(y\pi/r)/\sin((y \pm 1)\pi/r)$, the first few generators in the ABF basis for the Whitney representation for $2n-1=7$ may be written

$$U_1 = \begin{bmatrix}
2_- &&&&&&&&&&&&& \\
& 0 &&&&&&&&&&&& \\
&& 2_- &&&&&&&&&&& \\
&&& 0 &&&&&&&&&& \\
&&&& 2_- &&&&&&&&& \\
&&&&& 0 &&&&&&&& \\
&&&&&& 2_- &&&&&&& \\
&&&&&&& 0 &&&&&& \\
&&&&&&&& 0 &&&&& \\
&&&&&&&&& 0 &&&& \\
&&&&&&&&&& 0 &&& \\
&&&&&&&&&&& 2_- && \\
&&&&&&&&&&&& 0 & \\
&&&&&&&&&&&&& 0
\end{bmatrix}$$

$$U_2 = \begin{bmatrix}
1_+ & 1_+3_- &&&&&&&&&&&& \\
1 & 3_- &&&&&&&&&&&& \\
&& 1_+ & 1_+3_- &&&&&&&&&& \\
&& 1 & 3_- &&&&&&&&&& \\
&&&& 1_+ & 1_+3_- &&&&&&&& \\
&&&& 1 & 3_- &&&&&&&& \\
&&&&&& 1_+ & 1_+3_- &&&&&& \\
&&&&&& 1 & 3_- &&&&&& \\
&&&&&&&& 0 &&&&& \\
&&&&&&&&& 3_- && 1 && \\
&&&&&&&&&& 0 &&& \\
&&&&&&&&& 1_+3_- && 1_+ && \\
&&&&&&&&&&&& 0 & \\
&&&&&&&&&&&&& 0
\end{bmatrix}$$

$$U_3=\begin{bmatrix}
2_- & & & & & & & & & & & & & \\
& 0 & & & & & & & & & & & & \\
& & 0 & & & & & & & & & & & \\
& & & 2_+ & & & & & 2_+4_- & & & & & \\
& & & & 2_- & & & & & & & & & \\
& & & & & 0 & & & & & & & & \\
& & & & & & 0 & & & & & & & \\
& & & & & & & 2_+ & & & 2_+4_- & & & \\
& & & 1 & & & & & 4_- & & & & & \\
& & & & & & & & & 2_+ & & & 2_+4_- & \\
& & & & & & & 1 & & & 4_- & & & \\
& & & & & & & & & & & 0 & & \\
& & & & & & & & & 1 & & & 4_- & \\
& & & & & & & & & & & & & 0
\end{bmatrix}$$

$$U_4=\begin{bmatrix}
1_+ & & 1_+3_- & & & & & & & & & & & \\
& 1_+ & & 1_+3_- & & & & & & & & & & \\
1 & & 3_- & & & & & & & & & & & \\
& 1 & & 3_- & & & & & & & & & & \\
& & & & 1_+ & & 1_+3_- & & & & & & & \\
& & & & & 1_+ & & 1_+3_- & & & & & & \\
& & & & 1 & & 3_- & & & & & & & \\
& & & & & 1 & & 3_- & & & & & & \\
& & & & & & & & 0 & & & & & \\
& & & & & & & & & 0 & & & & \\
& & & & & & & & & & 0 & & & \\
& & & & & & & & & & & 0 & & \\
& & & & & & & & & & & & 3_+ & 3_+5_- \\
& & & & & & & & & & & & 1 & 5_-
\end{bmatrix}.$$

We may take any $n-1=3$ of these generators consecutively to give the regular representation of the three-operator algebra. Generically we see (from the first three) that the indecomposable structure is $2P_2 \otimes 3P_3 \otimes 1P_1$, where each d-dimensional projective P_d is also an irreducible module. Now addressing the case $r=2$ and using the operators U_2, U_3, U_4 we make similarity transformations of the form

$$\begin{bmatrix} y_+ & y_+(y+2)_- \\ 1 & (y+2)_- \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 \\ 1 & y_+ + (y+2) \end{bmatrix} \tag{13}$$

(note that $y_+ + (y+2)_- = \sqrt{q}$ for all y) whereupon we can allow $r \rightarrow 2$ and hence obtain

$$U_2=\begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & & 0 & & \\ & & 1 & 0 & \\ & & & & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & & 0 & & \\ & & 1 & 0 & \\ & & & & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

$$U_3=\begin{bmatrix}0&1&&&\\&0&&&\\&&0&1&\\&&&0&\\&&&&0\end{bmatrix}\oplus\begin{bmatrix}0&1&&&\\&0&&&\\&&0&1&\\&&&0&\\&&&&0\end{bmatrix}\oplus\begin{bmatrix}0&&&\\1&0&&\\&&0&\\&&&0\end{bmatrix}$$

$$U_4=\begin{bmatrix}0&&&&\\1&0&&&\\1&&0&&\\&1&1&0&2\\-1&&&&0\end{bmatrix}\oplus\begin{bmatrix}0&&&&\\1&0&&&\\1&&0&&\\&1&1&0&2\\-1&&&&0\end{bmatrix}\oplus\begin{bmatrix}0&1&&1\\&0&&\\&1&0&1\\&&&0\end{bmatrix}$$

i.e. $2P_5 \oplus P_4$ with simple content $2(2S_2 \oplus S_1) \oplus (2S_1 \oplus S_2)$.

In fact our arrival at the correct answer was somewhat fortuitous. Consider the generic regular representation of one operator

$$U_i=\begin{bmatrix}\sqrt{q}&\\&0\end{bmatrix}. \tag{14}$$

This may be similarity transformed to

$$U_i=\begin{bmatrix}\sqrt{q}&1\\&0\end{bmatrix} \tag{15}$$

whereupon $r \to 2$ gives the regular representation. However, if we put $r=2$ in the original version we obtain an inequivalent representation! The reason is that if we look at the similarity transformation *after* $r \to 2$ we find that it is singular. Care (or luck) is required to make these transformations in the right order.

Now the off-diagonal element gluing two copies of the irreducible representation together in (15) is missing in (14). Henceforward we will use the term 'glued' to describe such an indecomposable arrangement of irreducibles. In the present case the algebra is spanned by $\{1, U_i\}$ and the radical is $\{U_i\}$. The algebra quotiented by the radical here is the algebra with the additional relation $U_i=0$ (hence the irreducible representation), while the radical element is represented by an upper triangular matrix. In general we can always use the ABF representations to give us the irreducible representations of the algebra, even if (as above) we taken insufficient care to preserve the regular representation. A quick way to check that we have preserved this as well is by dimension counting at the end!

The other non-semisimple cases for three operators are $q=1$:

$$U_1=\sqrt{q}\,\mathrm{diag}(1,0,1,0,0,0,1,0,1,0,0,1,0,0)$$

$$U_3=\sqrt{q}\,\mathrm{diag}(1,0,0,1,0,1,0,0,0,1,0,1,0,0)$$

$$U_2=\begin{pmatrix}1&1\\0&0\end{pmatrix}\oplus\begin{pmatrix}1&1&1\\1&1&1\\-1&-1&-1\end{pmatrix}\oplus\begin{pmatrix}1&1&1\\1&1&1\\-1&-1&-1\end{pmatrix}$$

$$\oplus\begin{pmatrix}1&1&1\\1&1&1\\-1&-1&-1\end{pmatrix}\oplus\begin{pmatrix}1&1&0\\0&0&0\\1&1&0\end{pmatrix}$$

so the top is $M_3(\mathbb{C})\oplus\mathbb{C}\oplus\mathbb{C}$ (11 degrees of freedom) and the nilpotent part has composition law

$$(a,b,c)(x,y,z)=(0,0,bx) \qquad \text{(three degrees of freedom)}$$

and $q=2$:
$U_1=\sqrt{q}\,\mathrm{diag}(1,0,1,0,1,0,0,1,0,0,1,0,0,0)$
$U_3=\sqrt{q}\,\mathrm{diag}(1,0,1,0,0,1,0,0,1,0,0,1,0,0)$

$$U_2=1/\sqrt{q}\left[\begin{pmatrix}1&1\\1&1\end{pmatrix}\oplus\begin{pmatrix}1&1\\1&1\end{pmatrix}\oplus\begin{pmatrix}1&1&1\\1&1&1\\0&0&0\end{pmatrix}\oplus\begin{pmatrix}1&1&1\\1&1&1\\0&0&0\end{pmatrix}\oplus\begin{pmatrix}1&1&1&0\\1&1&1&0\\0&0&0&0\\1&1&1&0\end{pmatrix}\right]$$

i.e. the top is $M_2(\mathbb{C})\oplus M_2(\mathbb{C})\oplus\mathbb{C}$ (9 degrees of freedom) and the nilpotent part has composition law

$(a,b,c,d,e)(x,y,z,t,v)=(0,0,0,0,cx+dy)$ (five degrees of freedom).

The idempotents and other elementary operators exhibiting this structure are discussed later.

5. The structure of the algebra: representation theory approach

A convenient way to summarise the structure of the regular representation of $n-1$ operators is obtained as follows (we give a statement of this 'summary' without explanation in Martin and Westbury (1988) and Westbury (1988)).

The idea is to make the $m=0$ representation of $2n-1$ generators well defined for the value of r we want, and then to restrict to $n-1$ generators as above. The point of this procedure is that, although glue may be lost in similarity transformations like that involved in taking equation (14) to (15), irreducible content cannot be lost. By making *this* representation well defined we make well defined certain *irreducible* representations of algebras with number of generators between $n-1$ and $2n-1$ (see later). Since these *irreducible* representations are fully recovered, all the glue in representations obtained by restricting them to $n-1$ generators is recovered. Such irreducibles restrict to include a complete set of indecomposable projective representations at the $n-1$ generator level, and therefore all the glue is faithfully represented.

The number sequences $\{s_i\}$ appropriate for the $m=0$ representation of $2n-1$ generators are represented as paths on a lattice, e.g. with $n=10$ as in figure 1. The number of walks from $(1,1)$ to (j,s) on this lattice is

$$\binom{j-1}{h}-\binom{j-1}{h-1}\qquad\text{with}\qquad h=[(j-s)/2]\tag{16}$$

(where $\binom{n}{-1}=0$ (Jones 1983)). We arrange the set of walks from $(1,1)$ to $(2n+1,1)$ into subsets labelled by the last $n+1$ nodes. The first of these n is a node in the vertical centre line. We sort these centre nodes into node sets. The elements of a set

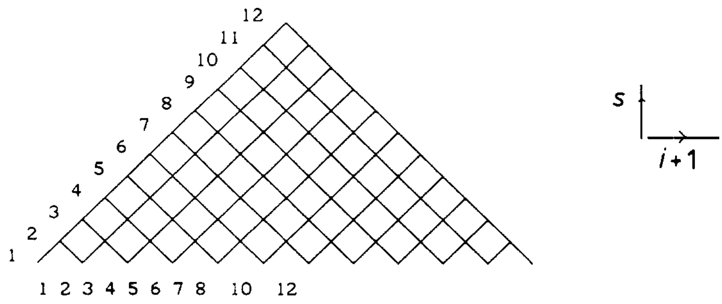


Figure 1. Positive walks from (1, 1) to (23, 23).

are mutual reflections in the various horizontal lines $s = mr$ ($m \in \mathbb{Z}+$), except for nodes on these lines, which each lie alone.

Generically the walk subsets correspond to bases for irreducible representations of the first $n-1$ operators, which are equivalent if and only if the centre node is the same. When $r \in \mathbb{Z}$ smaller irreducible subspaces may appear (as in Jones 1983), whereupon the generic irreducibles become glued pairs of irreducibles (in the sense of (15), see later). The cost of similarity transforms required to make matrix elements well defined is then to glue certain generic irreducibles (now glued pairs) in pairs. These similarity transforms also weaken the identification of walks with basis states. However we can still use the walk terminology as a convenient framework for our discussion.

The rules for the gluing are as follows. If the centre node is on a line then each generic irreducible remains irreducible and is not glued (see later). If no walk in the subset crosses the line above the centre node then the irreducible remains irreducible (the node is necessarily the top element of its node set). However, it is glued to a generic irreducible (glued pair) immediately below it in the same node set, if one exists. This latter is obtained by reflecting, in the line immediately below the node, the part of each walk between its closest crossings of the line on either side of the node. To get an idea of this, note that it is similarity transforms between such pairs of walks which are required to eliminate divergences, so they get mixed. The idea is that this mixing absorbs the lower generic irreducible into the current projective.

Other generic irreducibles develop invariant subspaces and thus become glued pairs. We name the irreducible on the invariant subspace after the centre node. The quotient subspace then gives, up to equivalence, the irreducible named after the node above in the appropriate node set. To see this, note that the quotient may be mapped to an appropriate subset of walks by reflection in a line. As before, the glued pair will also be glued to one immediately below it, if one exists. Note, for dimension counting purposes, that these lower generic irreducibles, once absorbed into a projective module, do not become further glued.

This procedure determines all the dimensions recursively. It is not a proof, but provides the organisational backbone for a proof to be discussed below. Note that the dimensions of the unitarisable irreducibles agree with those given in Jones (1983) or Martin (1987).

In any case, following this procedure the dimensions will always add up as required. For example, with $q=1$ ($r=3$) in the case above we have node sets $\{(12,12)\}$, $\{(12,10),(12,8),(12,4),(12,2)\}$, $\{(12,6)\}$. The corresponding projectives have irreducible content:

node:	$\{(12,12)\}$,	$\{(12,10)$,	$(12,8)$,	$(12,4)$,	$(12,2)\}$,	$\{(12,6)\}$
projective:	P_1	P_{54}	P_{209}	P_{297}	P_{132}	P_{110}
content:	1	10	34	131	1	110
		34	10 131	34 1	131	
		10	34	131		

The first 10 (in the top row of contents) is the dimension of the irreducible associated with the top node of the set of four. By our prescription this is still the generic dimension given by (16). If this is correct (as we will show in the appendix), then by the general theorem in the introduction it implies that the corresponding projective has multiplicity 10. The top 34 is thus the generic multiplicity for the corresponding module, 44 (from (16)), less 10 copies which have become absorbed in making the 10 copies of the previous projective. The other dimensions are then $131 = 165 - 34$ and $1 = 132 - 131$ by the same argument.

The projectives are arranged in the table above in such a way as to exhibit their Loewy decomposition (Benson 1984). That is, for each projective P the top row is the composition of P/JP, the next JP/J^2P, then J^2P/J^3P. Dimension counting (including glue) by summing over independent blocks in the projectives (there is at most one way of gluing one irreducible on top of another) the dimensions agree overall with the generic ones (and will obviously always do so if we follow this procedure). For example, in the case above we have independent blocks:

$$\begin{array}{cccc} [10.10] & & & \\ & \begin{bmatrix} 34.34 & 34.10 \\ 10.34 & 10.10 \end{bmatrix} & & \\ & & \begin{bmatrix} 131.131 & 131.34 \\ 34.131 & 34.34 \end{bmatrix} & \\ & & & \begin{bmatrix} 1.1 & 1.131 \\ 131.1 & 131.131 \end{bmatrix} \end{array} \tag{17}$$

We have arranged the blocks to show how the dimensions match the generic case. The top left-hand number in each block is the dimension associated with the top; the rest is in the radical.

We conclude this section with a technical remark for the more mathematically minded reader. The Bratelli diagram for the restrictions of irreducibles $\mu:(T_k(q)/J_k)\downarrow(T_{k-1}(q)/J_{k-1})$ (in the notation of Robinson 1961) may now be readily deduced, except for the restrictions of irreducibles associated with nodes on lines ($s = mr$, see above) which are awkward. In these cases the restriction is to the irreducible content of the projective associated with the node immediately above $s = mr$ in $T_{k-1}(q)$.

6. The structure of the algebra: operator approach

Returning to the idempotents of the algebra, we note that in the Beraha cases the normalisations κ_i may diverge and the generic construction break down. There are then two possibilities. Firstly the divergence may be cancelled by the vanishing of another factor. For example, with four operators, the last idempotent derived from I_1 is apparently not defined when $q = 0$ (I_1 itself is not defined in this case). However if we evaluate the idempotent as a function of q, expand the resultant and then put $q = 0$ we find that it is well defined. Indeed a complete set of elementary operators derived from I_1 may be found by appropriate similarity transformations on the formally derived idempotents. Arranging the operators to indicate their role in the corresponding matrix algebra we have

$$\begin{pmatrix} U_2U_3U_1U_4 & U_2U_3U_1U_2 & U_2U_3U_1 & U_2U_3U_1U_2U_4 & U_2U_3U_1A \\ U_4U_3U_1U_4 & U_4U_3U_1U_2 & U_4U_3U_1 & U_4U_3U_1U_2U_4 & U_4U_3U_1A \\ U_4U_2U_3U_1U_4 & U_4U_2U_3U_1U_2 & U_4U_2U_3U_1 & U_4U_2U_3U_1U_2U_4 & U_4U_2U_3U_1A \\ U_3U_1U_4 & U_3U_1U_2 & U_3U_1 & U_3U_1U_2U_4 & U_3U_1A \\ BU_3U_1U_4 & BU_3U_1U_2 & BU_3U_1 & BU_3U_1U_2U_4 & BU_3U_1A \end{pmatrix} \tag{18}$$

where $B = A^T = 1/\sqrt{-2}(U_3U_4U_2 - U_2 - U_4)$ and the well defined generic (i.e. original) idempotent is BU_3U_1A.

More generally we note that, although a single factor like (12) may not be defined at some r values, certain products of such factors can be well defined, or become well defined when acting on the idempotents they build on. We can think of these products as modifying the associated sequence or walk in more than just one step. In this case, if we can first find walks with idempotents which are defined then we can also have idempotents corresponding to modification by these well defined products. The further idea is that, as in (17) above, if an idempotent associated with a walk can be made well defined, then idempotents can be associated with every walk in its envelope (see the appendix).

Now the idempotent for the straight up walk 12345678 . . $(n+1)$, i.e. idem$[n]$, is well defined (after expansion) if $(n+1)=jr$ $(j \in \mathbb{Z}+)$ (see the appendix). Provided $q \neq 0$, I_m is thus well defined in all cases in which this idempotent is used. This provides a good supply of well defined idempotents to build on. The factor (12) has the effect of adding a 'diamond' to the previous walk, in the sense that the walk for the new idempotent is obtained from the old by, for example, figure 2.

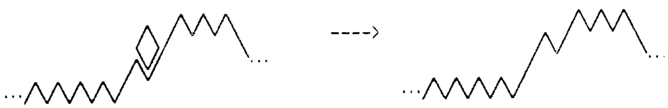


Figure 2. Adding a 'diamond' to a walk.

The factor is not defined if a line $s=jr$ cuts the diamond or touches its apex. The well defined products are the $r \times r$ diamonds whose corners each touch a line. For example the general 2×2 diamond of this kind (with $q=0$, $r=2$) is

$$\kappa_{jr-1}\kappa_{jr}^{3}\kappa_{jr+1}^{3}\kappa_{jr+2}\left(1-\frac{U_i}{\kappa_{jr+1}}\right)\left(1-\frac{U_{i-1}}{\kappa_{jr}}\right)\left(1-\frac{U_{i+1}}{\kappa_{jr}}\right)\left(1-\frac{U_i}{\kappa_{jr-1}}\right).$$

The only potentially divergent part of this factor is the coefficient of U_i, which is

$$-\kappa_{jr-1}\kappa_{jr}^{3}\kappa_{jr+1}^{3}\kappa_{jr+2}(\kappa_{jr+1}^{-1}+\kappa_{jr-1}^{-1}-\sqrt{q}(\kappa_{jr+1}\kappa_{jr-1})^{-1}+2(\kappa_{jr+1}\kappa_{jr}\kappa_{jr-1})^{-1}).$$

In fact the first factor here is $(j-1)/(j+1)$ and the second is zero.

Note that this means that we can construct a well defined idempotent associated with the last walk whenever $m+1=jr$ $(j \in \mathbb{Z}+)$. This is why we expect the corresponding representation to remain irreducible.

Alternatively the divergence may not cancel. In the case of I_m (and $q \neq 0$) the divergent part is then in the radical of the algebra, for which the overall normalisation is unimportant (in this sector renormalisation is a similarity transformation, so up to equivalence we may make arbitrary renormalisations (Anderson and Fuller 1974)). We may thus adopt the prescription of renormalising the divergent part, whereupon the construction may be resurrected. Note that a divergent operator is still formally idempotent, so the renormalised divergent part is nilpotent. For other primitive idempotents divergences may be in the radical, or otherwise cancel with those in further idempotents. This is the operator manifestation of the inappropriate choice of basis in the ABF representation, and is corrected by the similarity transformations discussed above. A rule of thumb is that cancellation is achieved by mixing idempotents corresponding to walks related by reflection in a line $s=jr$, between two crossings of that line. To see this, consider the effect of adding a diamond *across* a line. The procedure is more complicated if there are multiple crossings.

A precise account of this programme is given in the appendix. Here we will restrict ourselves to an illustrative example. With three operators and $q=1$ we have $m=0, 2, 4$. For $m=0$ we have

$$I_0 = U_1 U_3/q$$

(the configuration 12121) and, formally, another idempotent

$$\frac{(1-\sqrt{q}\,U_2)}{\sqrt{(q-1)}} I_0 \frac{(1-\sqrt{q}\,U_2)}{\sqrt{(q-1)}}$$

(the configuration 12321). The corresponding nilpotent renormalised operator

$$(1-\sqrt{q}\,U_2) I_0 (1-\sqrt{q}\,U_2)$$

and the nilpotent renormalised 'off-diagonal' operators

$$(1-\sqrt{q}\,U_2) I_0 \qquad \text{and} \qquad I_0(1-\sqrt{q}\,U_2)$$

together form a nilpotent double-sided ideal subalgebra (the maximal one, as it will turn out). On quotienting by the radical (i.e. setting these terms to zero), I_0 gives a one-dimensional double-sided ideal subalgebra. For $m=2$ we have generic idempotents

$$I_2 = U_1(1-q^{-1/2}U_3) \qquad (12123)$$

with

$$\frac{(1-\sqrt{q}\,U_2)}{\sqrt{q-1}} I_2 \frac{(1-\sqrt{q}\,U_2)}{\sqrt{q-1}} \qquad (12323)$$

and

$$\frac{(1-(q-1)U_3)}{\sqrt{q-2}} \frac{(1-\sqrt{q}\,U_2)}{\sqrt{q-1}} I_2 \frac{(1-\sqrt{q}\,U_2)}{\sqrt{q-1}} \frac{(1-(q-1)U_3)}{\sqrt{q-2}} \qquad (12343).$$

The sum of these last two is finite so, making an appropriate change of basis, these are replaced by

$$(I-U_3)(1-U_2)U_1(1-U_2)(1-U_3) \text{ and } U_3(1-U_2)U_1(1-U_2)U_3.$$

Altogether for $m=2$ we thus have a 3×3 dimensional double-sided ideal, as in the generic case.

Again quotienting by the radical the last idempotent, I_4, gives another one-dimensional double-sided ideal (we wrote out idem$_0$[4] explicitly in the previous section). The 14-dimensional algebra thus consists of an 11-dimensional semisimple 'top' and a three-dimensional radical. This should be compared with the corresponding representation theory example above.

With $q=2$ the $m=0$ generic operators remain well defined. The first two $m=2$ idempotents are well defined and, together with their off-diagonals form a four-dimensional double-sided ideal subalgebra after quotienting by the radical. This radical is given by renormalising the remaining operators to

$$I_2(1-\sqrt{2}\,U_2)(1-U_3)$$

and so on. Again this should be compared with the representation theory equivalent.

In this way the radical may be found, and the structure of the maximal semisimple quotient obtained, in each case. The irreducible components then label the possible long distance properties of any statistical mechanical model, as in the introduction. Specifically, there is an irreducible representation associated with each possible value of m (except for $r=2$, $m=0$ where the dimension is zero). The asymptotic rate of increase of dimension per lattice site for a given m is 4 unless $m<r-1$ (in which case the dimension is given by Martin (1986)).

7. Concluding remarks

We note the central role of the Whitney (or $m=0$) representation in the construction of the regular representation. This representation is the one responsible for the unique free energy in statistical mechanical models, i.e. it gives the largest eigenvalue of the transfer matrix for positive x. It is remarkable that such a physically important representation should also be the one which restricts, in the $2n-1$ generator case, to the regular representation of the $n-1$ generator subalgebra (which contains the entire spectrum of any possible model). This means, for example, that the spectrum of an appropriately inhomogeneously coupled $2n$-site wide lattice Potts model carries the entire spectrum (up to non-vanishing degeneracy) of all possible such models on an n-site wide lattice. The renormalisation group possibilities here are intriguing!

Acknowledgments

I would like to thank Bruce Westbury, Rob Wilson and Guy Launer for useful conversations, and the SERC for financial support during part of this work.

Appendix. Defining idempotents in the Beraha cases

In this appendix we will give results which allow us to construct elementary operators exhibiting the algebraic structures we have discussed.

(1) Idempotents of the form idem[kr − 1]

To show that idem$[jr-1]$ is well defined we proceed as follows. For notational convenience define $E_k = \text{idem}[k-1]$ and

$$W_{p,s} = 1 + u_p + u_p u_{p-1} + \ldots + u_p u_{p-1} \ldots u_s$$

where $u_p = -\kappa_{p-1} U_{p-2}$ and $W_{p,p+1} = 1$. Then with $t = p - s$ ($s \in \mathbb{Z}_+$) we find that

$$E_p = E_t \prod_{j=1}^{s} [W_{t+j,t+1} E_t] = E_{p-1} W_{p,3}$$

$$= E_t \prod_{j=1}^{s} [W_{t+j,t+1} W^T_{t,t-j+1}] E_t$$

where we have replaced E_t in the product by noting that all the discarded terms commute through to the left or right and are orthogonal to E_t.

Now if $E_{(k-1)r}$ is defined (E_r is manifestly so) then putting $d=(k-1)r$ and adopting the notation that in a product Π'_j the variable is incremented negatively, so that, for example, $\Pi_{j=2}^{1} u_j = 1$, but $\Pi_{j=2}^{\prime 1} u_j = u_2 u_1$, we have the following result for $c<r+1$:

$$X = \prod_{j=1}^{c}\left[\left(\prod_{a=d-j+2}^{d} u_a\right)\left(\prod_{b=d+j}^{\prime\, d+1} u_b\right)\right] W^T_{d,d-c+1} W_{d+c+1,d+1} E_d$$

$$= \prod_{j=1}^{c+1}\left[\left(\prod_{a=d-j+2}^{d} u_a\right)\left(\prod_{b=d+j}^{\prime\, d+1} u_b\right)\right] E_d .$$

To see this, note that

$$X = \prod_{j=1}^{c}\left[\left(\prod_{a=d-j+2}^{d} u_a\right)\left(\prod_{b=d+j}^{\prime\, d+1} u_b\right)\right]\left(W^T_{d,d-c+1}\prod_{b=d+c+1}^{\prime\, d+1} u_b + W_{d+c+1,d+2}\right) E_d$$

since $u_d E_d = 0$, and that $u_{d+2}=0$ and the remaining unwanted terms cancel summand by summand between the two series.

Every summand in every factor in E_{d+r} written as above is well defined, except for the term u_{d+1} in the first factor in the product. We must show that the action of this term on the terms on the right is to give a cancelling factor of κ_{r+1}. First note that we need only keep track of summands with finite coefficients (since there are no other divergent ones). We may then apply the result above with $c=1$. At the rth iteration on c we see that the required cancelling factor appears. This proves by iteration on k that such an idempotent E_{kr} is well defined.

(2) 'Big diamond' idempotents

First note that the product of factors corresponding to adding diamonds to a walk, as in figure 3, simplify greatly when acting on the idempotent I corresponding to the original walk in such cases (because of the orthogonality properties of this idempotent) to give the elementary operator $L_{i+1,j+1}(x)I$ where

$$L_{i+1,j+1}(x) = \left(\prod_{k=1}^{x}(\kappa_{j+k}\kappa_{j+k+1})\right)^{1/2}\left(1+\sum_{k=1}^{x}\prod_{m=k}^{\prime\, 1}(-U_{i+m}/\kappa_{j+m})\right). \qquad (A1)$$

This is well defined provided I is well defined and, with $k, k' \in \mathbb{Z}_+$ and $k' \geqslant k$, $(k-1)r < j \leqslant kr-2$ and $k'r \leqslant j+x < (k'+1)r-2$ (even though the full product of factors is not well defined in general).

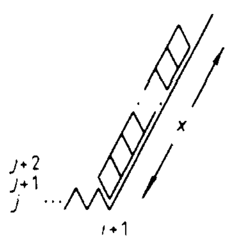


Figure 3. Adding a row of diamonds to a walk.

The products of factors corresponding to 'big' (i.e. $r\times r$) diamonds may then be shown to be well defined (when acting on the appropriate idempotents), by building them up in strips using (A1). The proof is non-trivial, since the first strip required violates the limits above and is not itself well defined. However, all the subsequent strips are well defined, so it is possible to keep track of the divergent terms. The divergence vanishes on the application of the rth (and final) strip.

Using these observations we can construct well defined operators associated with the irreducible representations of the TL algebra for any r. The procedure is as follows. The initial idempotent is made well defined by discarding its divergent part (quotienting by the radical). We may then construct further well defined operators by adding diamonds until the walk $(12)^k 34..2r..r..2r..r..2r.....r..(m+1)$ is achieved (where $k<r+1$). To see that this is well defined, consider the walk $1234..2r..r$, which gives a well defined operator from which the generalisations may be constructed. We may then add as many big diamonds as possible as in figure 4.

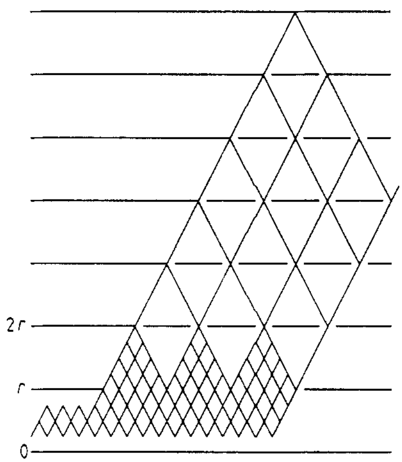


Figure 4. Adding 'big' diamonds to a walk.

Small diamonds may then be added to the lower left of this picture until there are no more places for them to go on the left of the main peak. We will show how the operators within the envelope so produced are made well defined in the next section. We then take the initial idempotent and add the longest strip possible of the kind shown in figure 3. Note that the resultant walk is not in the envelope of the above construction. We then repeat the above construction with this new starting shape. We then add another strip and repeat, and so on. Provided we can make all the operators in the envelope well defined we then have an idempotent for each walk which does not touch the line above $m+1$ after it has last touched the line below $m+1$. The number of such walks is clearly the dimension of the corresponding irreducible representation (consider the regular representation construction, and turn the diagram on its side!). A simple reflection argument also shows that the walks not accounted for in this scheme are in one-to-one correspondence with counted walks for the representation immediately above this one in the node set. The unaccounted walks do touch the line above after they last touch the line below. The reflection is to replace the steps after the last touching of the line above with their image in this line. This automatically takes the endpoint $m+1$ to the point above it in the node set for each of these walks ($m'+1$, say). Because none of the original walks touches the line below $m+1$ after it touches the line above, the reflection never touches the line above $m'+1$ after it last crosses the line below $m'+1$. All walks with this property are generated, but these are precisely the walks counted in the original construction to $m'+1$. The argument is as follows (generalising from Martin (1986), where the conclusion is wrongly stated!).

First note that no two distinct walks reflect to the same walk. Then note that this is also true for the reverse reflection from counted walks in the case above. It is a simple combinatorial exercise to show that the dimensions add up (see, for example, Westbury 1988).

The interpretation of the operators associated on the left with counted and on the right with uncounted walks (and vice versa) is that they cover the part of the radical gluing the corresponding two irreducible representations together (see the example of a Loewy decomposition given in the text). The operators associated on the left and right with uncounted walks glue together two copies of the irreducible above in the node set. We see that the whole structure is accounted for in this way, the dimensions being arranged as in (17).

(3) Other operators

It remains to show that every operator in the envelope of the well defined operators may be made well defined. Without loss of generality we can consider building up from the initial position shown in figure 5, which we may take to have a well defined idempotent associated (call it I). Then adding a diamond in the ith position we have

$$N = \kappa_{kr}\kappa_{kr-1}(1 - U_i/\kappa_{kr-1})I(1 - U_i/\kappa_{kr-1}) \tag{A2}$$

where κ_{kr} is divergent. Adding a diamond in the $(i+1)$th position we then get

$$\kappa_{kr}\kappa_{kr+1}(1 - U_{i+1}/\kappa_{kr})N(1 - U_{i+1}/\kappa_{kr}) \tag{A3}$$

where $\kappa_{kr}\kappa_{kr+1}$ is well defined. We write these formal idempotents out in a matrix, together with their off-diagonal mixings, and make linear transformations along them which preserve the trace:

$$\begin{bmatrix} N & \sqrt{(\kappa_{kr}\kappa_{kr+1})}N(1 - U_{i+1}/\kappa_{kr}) \\ \sqrt{(\kappa_{kr}\kappa_{kr+1})}(1 - U_{i+1}/\kappa_{kr})N & \kappa_{kr}\kappa_{kr+1}(1 - U_{i+1}/\kappa_{kr})N(1 - U_{i+1}/\kappa_{kr}) \end{bmatrix}$$
$$\to \begin{bmatrix} U_{i+1}(N/\kappa_{kr})U_{i+1} & \mathrm{i}U_{i+1}(N/\kappa_{kr})(\sqrt{q}(1 - U_{i+1}/\sqrt{q})) \\ \mathrm{i}(1 - U_{i+1}/\sqrt{q})(N/\kappa_{kr})U_{i+1} & -(1 - U_{i+1}/\sqrt{q})(N/\kappa_{kr})(\sqrt{q}(1 - U_{i+1}/\sqrt{q})) \end{bmatrix}. \tag{A4}$$

Now since each of the formal idempotents separately was orthogonal to all other primitive idempotents, these linear combinations will be similarly orthogonal. Since

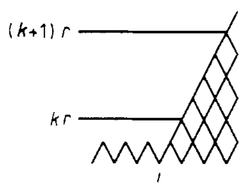


Figure 5. The walk associated with idempotent I.

these new objects are also idempotent, mainfestly mutually orthogonal and well defined, they are the objects we want to replace the undefined versions. Similarly the four idempotents shown in figure 6 must be mixed. Each has a double divergence. Mixing them in pairs eliminates one divergence as above, mixing the pairs in the same way (i.e. cross producting copies of the above transformation) then eliminates the remaining divergence. Note that these walks may be seen as reflections of one another in the line. By the same measure the pair shown in figure 7(a) must be mixed, the pair in figure 7(b) must be mixed, and so on. It is easy to check using (A1) that the divergences cancel between the two in figure 7(a), for example, the first is

$$M = \kappa_r \kappa_{r-1}(1 - U_{i-2}/\kappa_{r-1}) I (1 - U_{i-2}/\kappa_{r-1})$$

with an overall κ_r divergence, and the second is

$$\begin{aligned} &\kappa_{r+2}\kappa_{r+1}(\kappa_{r+1}\kappa_r)^2 \kappa_r \kappa_{r-1}(1 - U_{i-1}/\kappa_r + U_i U_{i-1}/\kappa_{r+1}\kappa_r)(1 - U_i/\kappa_{r-1} \\ &\qquad + U_{i+1} U_i/\kappa_r \kappa_{r-1}) M (1 - U_i/\kappa_{r-1} + U_{i+1} U_i/\kappa_r \kappa_{r-1}) \\ &\qquad \times (1 - U_{i-1}/\kappa_r + U_i U_{i-1}/\kappa_{r+1}\kappa_r) \end{aligned}$$

where $\kappa_{r+2}\kappa_{r-1} = 1$ and $\kappa_{r+1}\kappa_r = -1$ from the definition, cancellation then occurs on application of the TL relations.

The construction of well defined operators within big diamonds proceeds in the same way. For example, the mixings in a 3×3 diamond are shown in figure 8.

Up to some trivial variations for dealing with the envelope of non-big-diamond walks on the left of figure 4, this completes the argument.

(4) Contributions to the radical

We may check that the radical glues together irreducibles in the reflection pattern described in the text as follows. Consider the formal idempotent for the top walk in

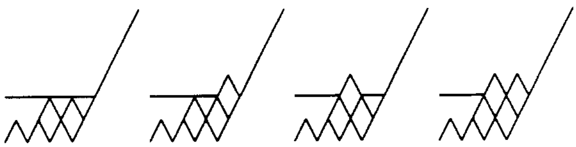
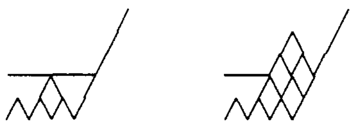
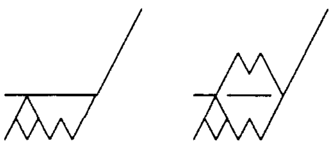


Figure 6. Walks obtained from I.



(*a*)



(*b*)

Figure 7. More walks obtained from I.

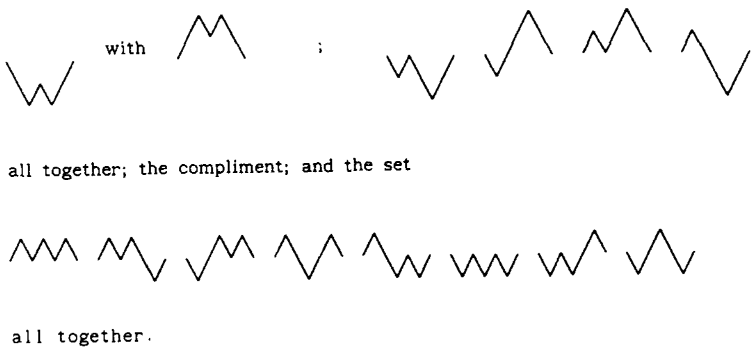


Figure 8. Walks within a big diamond.

figure 9(a) which is (from (A1), and with $E = \mathrm{idem}_2[kr-2]$)

$$\kappa_{kr}A = \kappa_{kr}\kappa_{kr-1}\left(1-\frac{U_{kr-1}}{\kappa_{kr-1}}\right) L_{2,2}(kr-3)\frac{U_1}{\sqrt{q}}\, E\, L^T_{2,2}(kr-3)\left(1-\frac{U_{kr-1}}{\kappa_{kr-1}}\right)$$

$$= L_{2,2}(kr-2)\frac{U_1}{\sqrt{q}}\, E\, L^T_{2,2}(kr-2).$$

We note that this is well defined apart from an overall factor of κ_{kr} and any divergences in U_1E (see later). Now consider the 'reflection' in $s = kr$, the walk in figure 9(b) which is (using the $U_i \to U_{b-i}$ invariance of idem[b])

$$\mathrm{idem}[kr] = \mathrm{idem}_1[kr-1](1-\kappa_{kr}U_1)\,\mathrm{idem}_1[kr-1].$$

The only divergence here is in the term containing $-\kappa_{kr}U_1$. Now $\kappa_{r+a} = \kappa_a$ and $\kappa_{-a} = 1/\kappa_{a+1}$, so we have

$$\mathrm{idem}_1[kr-1] = \left(1-\frac{U_{kr-1}}{\kappa_{kr-1}}\right) L_{2,2}(kr-3)\, E\left(\prod_{j=1}^{kr-3}(\kappa_{1+j}\kappa_{2+j})^{1/2}\right)^{-1}$$

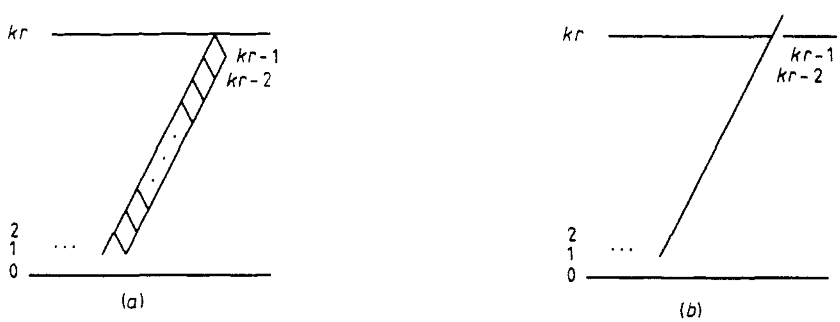


Figure 9. A walk associated with the radical, and its 'reflection'.

where the denominator is just $(\kappa_2\kappa_{kr-1})^{1/2}=1$. The divergent part of idem$[kr]$ is thus $-\kappa_{kr}A$. Since idem$[kr-1]$ is finite, A is finite as required and the κ_{kr} divergence is the worst divergence overall. We see that the divergences here and above are equal and opposite. We build further walks from figures 9(a) and (b) prefixed by the walk 121212..121 in identical ways (hence continued cancellations) until the new walks next touch the line $s = kr$ (a situation we have already discussed). If we write idem$[kr]=-\kappa_{kr}A+B$ (A, B well defined) then since, formally, idem$[kr]^2=$ idem$[kr]$ we have $A^2=0$, and similarly $U_iA=0$ $(i=1,\ldots,kr)$ and so on. Thus A is a double-sided nilpotent ideal in $T_{kr-1}(q)$. Furthermore, on putting $A=0$ we find $B^2=B$, $U_iB=0$ and so on. This procedure may be generalised to other reflections to identify the rest of the radical.

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